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Knizhnik-Zamolodchikov equations and spectral flow in AdS_3 string theory

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Abstract: I generalize the Knizhnik-Zamolodchikov equations to correlators of spectral flowed fields in AdS_3 string theory. If spectral flow is preserved or violated by one unit, the resulting equations are equivalent to the KZ equations. If spectral flow is violated by two units or more, only some linear combinations of the KZ equations hold, but extra equations appear. Then I explicitly show how these correlators and the associated conformal blocks are related to Liouville theory correlators and conformal blocks with degenerate field insertions, where each unit of spectral flow violation removes one degenerate field. A similar relation to Liouville theory holds for noncompact parafermions.

Keywords: Conformal Field Models in String Theory.

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Contents

1. Introduction and overview

The Knizhnik-Zamolodchikov equations [1] are an essential tool in the study of conformal field theories with affine Lie algebra symmetry.[1] All correlation functions of affine primary fields obey this system of linear differential equations, which determine their dependence on worldsheet coordinates.

However, in string theory in AdS_3, whose associated conformal field theory has an affine $\mathrm{SL}(2,\mathbb{R})$ symmetry, Maldacena and Ooguri have shown that the physical spectrum cannot be built only from affine primaries [3]. Instead, one should also include spectral-flowed fields. Correlation functions involving such fields are not expected to obey the KZ equations.

Nevertheless, such spectral-flowed fields are obtained from affine primaries via the spectral flow automorphism of the affine Lie algebra. This will enable me to derive generalized KZ equations for their correlation functions. The other main purpose of this work is to explicitly relate these correlation functions to Liouville theory correlations functions. This amounts to solving the generalized KZ equations in terms of Virasoro conformal blocks.

[1] For a review of original work on such theories, see [2] and references therein. In particular, early works on spectral flow are referenced in section 2.1 of that article.

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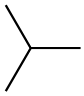
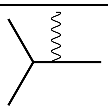
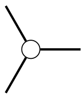
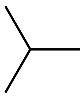
Let me now sketch the results. If a correlation function respects spectral flow conservation, then it will satisfy a system of equations (2.18) which turns out to be equivalent to the KZ equations via a simple twist.[2] (Actually, this conclusion also holds if spectral flow conservation is violated by one unit, due to the global group symmetry.) If a correlation function violates spectral flow conservation, then it will satisfy only some specific linear combinations (2.28) of the KZ equations. However, the missing equations will be replaced with simpler constraints (2.30) which do not involve derivatives wrt worldsheet coordinates.

Therefore, the equations obeyed in the case when spectral flow is not conserved are in some sense simpler than the original KZ equations. This will become clear after I show how to perform Sklyanin's separation of variables for such equations. Each unit of spectral flow violation leads to the disappearance of one variable-separated equation, until there are none left in the case of maximal violation.

These variable-separated equations are actually identical to Belavin-Polyakov-Zamolodchikov equations (2.41). I will exploit this in order to derive a relation between correlation functions of n spectral-flowed fields in the H_3^+ model and correlation functions in Liouville theory. (The H_3^+ model, or string theory in the euclidean AdS_3, is introduced here for technical reasons.) If spectral flow conservation is violated by r units, the relevant Liouville correlation functions will have $n-2-r$ degenerate field insertions (3.26). This shows why the maximal spectral flow violation $n-2$ is equal to the number of Liouville degenerate fields needed to reproduce an unflowed H_3^+ correlator. A similar relation with Liouville theory also holds for the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset model (3.29).

Deriving such relations between correlation functions involves not only the KZ equations, but also the structure constants of the H_3^+ model. The ordinary, spectral flow-preserving structure constant is known to be equal to the Liouville structure constant associated with a Liouville vertex dressed with one degenerate field [5]. Here I will show that the H_3^+ spectral flow-violating vertex corresponds to an ordinary Liouville vertex, at the levels of structure constants (3.15) (3.19), operator product expansions (3.24) (3.25), and conformal blocks (3.36). In particular, I will argue for the existence of a spectral flow-violating operator product expansion in the H_3^+ model, as an alternative to the ordinary operator product expansion. This is shown schematically in the diagrams below.

euclidean AdS_3		Liouville theory	
Flow − preserving vertex			Vertex dressed with one degenerate field
Flow − violating vertex			Ordinary vertex

(1.1)

In an outlook, I will mention possible applications of these results to string theory in AdS_3 and to the definition of a fusing matrix for the H_3^+ model.

[2] Such twisted KZ equations are already known in the context of WZW orbifolds [4].

2. KZ equations and spectral flow

In this section I derive which modifications of the KZ equations apply to correlation functions involving spectral flowed fields.

2.1 Preliminaries: definition of the spectral flowed fields

Let me consider a conformal field theory with an chiral affine Lie algebra symmetry $\widehat{s\ell_2}$ at level $k > 2$, living on the Riemann sphere parametrized by complex coordinates $z, \bar{z}$. In this section, I will be concerned only with the holomorphic sector. The symmetry of this sector is a "left-moving" copy of the algebra $\widehat{s\ell_2}$,

$$\begin{cases} [J^3_n, J^3_m] = -\frac{k}{2} n \delta_{n+m,0} \,, \\ [J^3_n, J^\pm_m] = \pm J^\pm_{n+m} \,, \\ [J^+_n, J^-_m] = -2J^3_{n+m} + kn\delta_{n+m,0} \,. \end{cases} \tag{2.1}$$

The generators J^a_n can be encoded in holomorphic currents $J^a(z)$:

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n \,, \qquad J^a_n = \frac{1}{2\pi i} \oint_0 dz\ z^n J^a(z) \,, \tag{2.2}$$

where $\oint_0$ stands for the integral along a contour encircling the point $z = 0$. The conformal symmetry generators L_n are built from the $\widehat{s\ell_2}$ generators via the standard Virasoro construction, which yields the central charge $c = \frac{3k}{k-2}$.

The spectral flowed field $\Phi^{j,w}(z)$ of spin j and spectral flow number $w \in \mathbb{Z}$ is defined as a primary with respect to the spectral flowed currents $\tilde{J}(z)$. These currents can be defined via their modes $\tilde{J}^a_n$, which are then used to build a spectral flowed copy $\tilde{L}_n$ of the Virasoro algebra [3]:

$$\begin{cases} \tilde{J}^3_n = J^3_n - \frac{k}{2} w \delta_{n,0} \,, \\ \tilde{J}^\pm_n = J^\pm_{n\pm w} \,, \\ \tilde{L}_n = L_n + wJ^3_n - \frac{k}{4} w^2 \delta_{n,0} \,. \end{cases} \tag{2.3}$$

Namely, the state $|j,w\rangle$ corresponding to the field $\Phi^{j,w}(z)$ is assumed to obey

$$\begin{cases} \tilde{J}^a_{n>0} |j,w\rangle = 0 \,, \\ \tilde{J}^a_0 |j,w\rangle = -t^a |j,w\rangle \,. \end{cases} \tag{2.4}$$

Equivalently, the field $\Phi^{j,w}(z)$ has the following operator product expansion with the *ordinary* currents $J^a(z)$:

$$\begin{cases} J^3(z)\Phi^{j,w}(y) \sim \frac{-t^3 \Phi^{j,w}(y)}{z-y} + \frac{kw}{2} \frac{\Phi^{j,w}(y)}{z-y} \,, \\ J^+(z)\Phi^{j,w}(y) \sim \frac{-t^+ \Phi^{j,w}(y)}{(z-y)^{1+w}} \,, \\ J^-(z)\Phi^{j,w}(y) \sim \frac{-t^- \Phi^{j,w}(y)}{(z-y)^{1-w}} \,. \end{cases} \tag{2.5}$$

Here t^a are generators of the $s\ell_2$ algebra. The field $\Phi^{j,w}(z)$ indeed carries a representation of $s\ell_2$ of spin j and Casimir $\frac{1}{2}(t^+t^- + t^-t^+ - 2t^3t^3) = -j(j+1)$, although the corresponding degrees of freedom are not spelt out explictly so far. I will later assume that $\Phi^{j,w}(z)$ belongs to a principal continuous series representation with spin $j \in -\frac{1}{2} + i\mathbb{R}$, whose states can be labelled using a complex parameter μ such that

$$\begin{cases} t^+ = \mu\,, \\ t^3 = \mu\frac{\partial}{\partial\mu}\,, \\ t^- = \mu\frac{\partial^2}{\partial\mu^2} - \frac{j(j+1)}{\mu}\,. \end{cases} \tag{2.6}$$

Another basis for the continuous representation is obtained by diagonalizing t^3 with eigenvalue $-m$ and considering $t^\pm$ as raising and lowering operators. Then the state $|j,w,m\rangle$ corresponding to the field $\Phi^{j,w}_m(z)$ satisfies

$$\tilde{J}^3_0 = -t^3 = J^3_0 - \frac{kw}{2} = m\,. \tag{2.7}$$

An advantage of the m-basis fields is that they happen to be eigenvalues of the original dilatation operator L_0 and therefore scale as follows:

$$\begin{aligned} \left(z\frac{\partial}{\partial z} + \Delta^{j,w}_m\right)\Phi^{j,w}_m(z) &= 0\,, \\ \Delta^{j,w}_m &= \Delta_j - wm - \frac{k}{4}w^2\,, \\ \Delta_j &= -\frac{j(j+1)}{k-2}\,. \end{aligned} \tag{2.8}$$

Moreover, the m-basis fields are simply related to parafermionic fields Ψ^j_m of the coset model $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ [3],

$$\Phi^{j,w}_m = e^{i\left(m+\frac{k}{2}w\right)\sqrt{\frac{2}{k}}\phi(z)}\Psi^j_m\,, \tag{2.9}$$

where $\phi(z)$ is a free boson such that $J^3(z) = -i\sqrt{\frac{k}{2}}\partial_z\phi(z)$.

From the relation with parafermions (2.9), it may seem easy to compute the correlation function of n spectral flowed fields and to determine the differential equations it satisfies, by relating it to a correlation function with no spectral flow. The only dependence on the spectral flow w is indeed in the free boson factor. However, the corresponding free boson correlation function does make sense only if the total charge vanishes, $\sum_{i=1}^n \left(m_i + \frac{k}{2}w_i\right) = 0$. In the case with no spectral flow $w_i = 0$, this implies $\sum_{i=1}^n m_i = 0$. In the case with spectral flow, the last two equalities imply spectral flow conservation $\sum_{i=1}^n w_i = 0$. Therefore, only spectral flow-preserving correlators are related to correlators without spectral flow. I will use this in order to check the KZ-type equations which I will derive for them.

2.2 KZ-type equations for spectral flow-preserving correlators

Each of the n KZ equations determines the dependence of a correlation function with respect to the worldsheet position of one field z_i. This is done by inserting the worldsheet translation operator L_{-1},

$$L_{-1}\Phi^{j_i,w_i}(z_i) = \frac{\partial}{\partial z_i}\Phi^{j_i,w_i}(z_i)\,. \tag{2.10}$$

The next step is to express L_{-1} in terms of the currents J^a. Having better control on the action of the spectral flowed currents $\tilde{J}^a$ on the spectral flowed field $\Phi^{j,w}(z)$, it is actually more convenient to use

$$\left((k-2)\tilde{L}_{-1} + \tilde{J}^a_{-1}\tilde{J}^a_0\right)\Phi^{j_i,w_i}(z_i) = 0\,, \tag{2.11}$$

and to rely on equation (2.3) to relate $\tilde{L}_{-1}$ to the translation operator L_{-1}. The resulting equation is:

$$\left[(k-2)\frac{\partial}{\partial z_i} + 2\tilde{J}^3_{-1}\left(-t^3_i + \frac{k-2}{2}w_i\right) + \tilde{J}^+_{-1}t^-_i + \tilde{J}^-_{-1}t^+_i\right]\Phi^{j_i,w_i}(z_i) = 0\,. \tag{2.12}$$

Now let me insert this null-vector equation into an n-point correlation function. The operators $\tilde{J}^a_{-1}$ have to be expressed in terms of Lie algebra generators t^a acting on the other fields $\Phi^{j_\ell,w_\ell}(z_\ell)$ for $\ell \neq i$. In the case of J^+, this is possible thanks to the equation:

$$\left\langle \frac{1}{2\pi i}\oint_\infty \frac{dz}{z-z_i}\prod_{\ell=1}^n (z-z_\ell)^{w_\ell} J^+(z)\prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0\,. \tag{2.13}$$

This equation holds provided $\sum_{\ell=1}^n w_\ell \leq 0$. This indeed implies that the function $\prod_{\ell=1}^n (z-z_\ell)^{w_\ell}$ is bounded near $z=\infty$ and allows closing the contour there, knowing $J^+(z) \sim \frac{1}{z}$.

Starting with equation (2.13), the contour of integration can be contracted into small loops around each point z_ℓ. With the help of the operator product expansion $J^a(z)\Phi^{j,w}(y)$ (2.5), this yields:

$$\left[\rho_i J^+_{-1,i} - \rho_i\sum_{\ell\neq i}\frac{w_\ell}{z_i - z_\ell}t^+_i - \sum_{\ell\neq i}\frac{\rho_\ell}{z_\ell - z_i}t^+_\ell\right]\left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0\,, \tag{2.14}$$

where the index i in $J^+_{-1,i}$ and t^+_i indicates which field they act on, and

$$\rho_i \equiv \prod_{\ell\neq i}(z_i - z_\ell)^{w_\ell}\,. \tag{2.15}$$

Similar manipulations are possible with J^- provided $\sum_\ell w_\ell \geq 0$, and yield:

$$\left[\rho_i^{-1} J^-_{-1,i} + \rho_i^{-1}\sum_{\ell\neq i}\frac{w_\ell}{z_i - z_\ell}t^-_i - \sum_{\ell\neq i}\frac{\rho_\ell^{-1}}{z_\ell - z_i}t^-_\ell\right]\left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0\,. \tag{2.16}$$

In the case of J^3, the following equation does not require any constraint on w_ℓ:

$$\left[J^3_{-1,i} - \sum_{\ell\neq i}\frac{1}{z_\ell - z_i}\left(t^3_\ell - \frac{k}{2}w_\ell\right)\right]\left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0. \tag{2.17}$$

In the spectral flow-preserving case $\sum_{\ell=1}^n w_\ell = 0$, the three equations (2.14), (2.16), (2.17) hold. Plugging them into equation (2.12) yields the following generalization of the KZ equations:[3]

$$\tilde{\mathcal{E}}_i \left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell) \right\rangle = 0 \qquad \text{with} \qquad \tilde{\mathcal{E}}_i \equiv \left((k-2)\frac{\partial}{\partial z_i} + \sum_{j\neq i} \frac{\tilde{Q}_{ij}}{z_j - z_i} \right), \tag{2.18}$$

$$\tilde{Q}_{ij} \equiv -2t_i^3 t_j^3 + t_i^- t_j^+ \frac{\rho_j}{\rho_i} + t_i^+ t_j^- \frac{\rho_i}{\rho_j} + (k-2)(w_j t_i^3 + w_i t_j^3) - \frac{k(k-2)}{2} w_i w_j \,. \tag{2.19}$$

These equations are related to the ordinary KZ equations $\mathcal{E}_i = 0$ by a twist of the correlation function:

$$\mathcal{E}_i \kappa^{-1} \left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell) \right\rangle = 0, \tag{2.20}$$

with

$$\mathcal{E}_i \equiv \left((k-2)\frac{\partial}{\partial z_i} + \sum_{j\neq i} \frac{Q_{ij}}{z_j - z_i} \right), \qquad Q_{ij} \equiv -2t_i^3 t_j^3 + t_i^- t_j^+ + t_i^+ t_j^-, \tag{2.21}$$

$$\kappa \equiv \prod_{j<i} (z_j - z_i)^{w_j t_i^3 + w_i t_j^3 - \frac{k}{2} w_i w_j} \,. \tag{2.22}$$

This is shown by a direct computation which uses the spacetime SL(2) invariance of the vacuum,

$$\left\langle \frac{1}{2\pi i} \oint_\infty dz\ J^3(z) \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell) \right\rangle = 0 \quad \Rightarrow \quad \sum_{i=1}^n \left(t_i^3 - \frac{k}{2} w_i \right) = 0 \quad \Rightarrow \quad \sum_{i=1}^n t_i^3 = 0 \,. \tag{2.23}$$

A check of the equation (2.21) can be performed using the relation of spectral flowed fields to parafermionic fields (2.9). This relation implies that the correlation function with spectral-flowed fields $\left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell) \right\rangle$ is equal to an unflowed correlator up to free boson correlators:

$$\left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(z_\ell) \right\rangle = \frac{\left\langle \prod_{\ell=1}^n e^{i\left(m_\ell + \frac{k}{2} w_\ell\right)\sqrt{\frac{2}{k}}\phi(z_\ell)} \right\rangle}{\left\langle \prod_{\ell=1}^n e^{i m_\ell \sqrt{\frac{2}{k}}\phi(z_\ell)} \right\rangle} \left\langle \prod_{\ell=1}^n \Phi^{j_\ell}(z_\ell) \right\rangle = \kappa \left\langle \prod_{\ell=1}^n \Phi^{j_\ell}(z_\ell) \right\rangle, \tag{2.24}$$

where m_ℓ is by definition the eigenvalue of $-t_\ell^3$.

Therefore, the spectral flow-preserving correlators satisfy the ordinary KZ equations modulo a simple twist. I will now generalize the methods used to derive these equations to the spectral flow-violating case. The same twist will provide notable simplifications of the equations, without reducing them to the ordinary KZ equations.

[3]These equations are a special case of the twisted KZ equations of [4]. Meanwhile, I believe the generalized KZ equations in the next subsection are new.

2.3 KZ-type equations for spectral flow-violating correlators

Consider an n-point correlator which violates spectral flow by $r \geq 1$ units, say $\sum_{\ell=1}^{n} w_\ell = -r$. The reasoning of the previous subsection which led to KZ-type equations now fails because equation (2.16), which expressed the action of J^-_{-1} on a field in terms of the action of t^- on the other fields, no longer holds. To derive such an equation would require

$$\frac{1}{2\pi i}\oint_\infty \frac{dz}{z-z_i}\prod_{\ell=1}^{n}(z-z_\ell)^{-w_\ell}\ J^-(z) \stackrel{?}{=} 0\,, \tag{2.25}$$

where the l.h.s. behaves near $z=\infty$ as $\frac{1}{2\pi i}\oint_\infty dz\ z^{r-2}$.

Actually, in the case $r=1$, the spacetime SL(2) symmetry of the vacuum is able to save the day. This symmetry indeed reads

$$\frac{1}{2\pi i}\oint_\infty dz\, J^a(z) = 0\,, \tag{2.26}$$

which implies eq. (2.25). But, for $r \geq 2$, it is impossible to derive n equations governing the z_i dependence of the correlators. Instead of equation (2.25), it is however possible to use the weaker equations:

$$\frac{1}{2\pi i}\oint_\infty \frac{dz}{\prod_{\alpha=1}^{r}(z-z_{i_\alpha})}\prod_{\ell=1}^{n}(z-z_\ell)^{-w_\ell}\, J^-(z) = 0\,, \tag{2.27}$$

for any choice of r distinct indices $i_1, i_2, \ldots, i_r$. This leads to an expression for a linear combination of $J^-_{-1,i_1}, J^-_{-1,i_2}, \ldots, J^-_{-1,i_r}$ in terms of $t^-_\ell, \ell = 1, \ldots, n$. Then it is possible to derive a differential equation for the spectral flow-violating n-point correlator, whose z-derivative part is a linear combination of $\frac{\partial}{\partial z_{i_1}}, \frac{\partial}{\partial z_{i_2}}, \frac{\partial}{\partial z_{i_r}}$. It is not necessary to go into much detail here: these manipulations actually also hold in the spectral flow-preserving case, and they can therefore yield nothing but a linear combination of the KZ-type equations (2.18) which hold in that case. The actual combination can easily be read from eq. (2.27),

$$\tilde{\mathcal{E}}_{\{i_\alpha\}} \equiv \sum_{\alpha=1}^{r}(\rho_{i_\alpha} t^+_{i_\alpha})^{-1}\frac{1}{\prod_{\beta\neq\alpha}(z_{i_\alpha}-z_{i_\beta})}\tilde{\mathcal{E}}_{i_\alpha} \qquad \text{for all } \{i_1, i_2, \ldots, i_r\} \subset \{1, 2, \ldots, n\}, \tag{2.28}$$

where ρ_i was defined in eq. (2.15). That only such combinations of r equations hold, means that $r-1$ KZ equations have been lost because of spectral flow violation $\sum_{\ell=1}^{n} w_\ell = -r$. This was because less equations could be obtained from the $J^-(z)$ current. Conversely, it is now possible to obtain new equations from the $J^+(z)$ current, using

$$\oint_\infty dz\, \prod_{\ell=1}^{n}(z-z_\ell)^{w_\ell}\, z^j J^+(z) = 0\,, \qquad j = 0, 1 \ldots, r\,. \tag{2.29}$$

This results in $r+1$ equations,

$$\sum_{\ell=1}^{n} z_\ell^j \rho_\ell t^+_\ell \left\langle \prod_{\ell=1}^{n} \Phi^{j_\ell, w_\ell}(z_\ell)\right\rangle = 0\,. \tag{2.30}$$

The $j = 0$ equation already held in the spectral flow-preserving case as a consequence of the spacetime SL(2) symmetry. The other equations, however, are specific to the spectral flow-violating case.

To conclude this subsection, let me gather the equations satisfied by the spectral flow-violating correlators, while simplifying them by applying the twist by the function κ (2.22),

$$\mathcal{E}_{\{i_\alpha\}}\,\kappa^{-1}\left\langle\prod_{\ell=1}^{n}\Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0 \qquad \text{for} \quad \{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,n\}\,, \tag{2.31}$$

$$\sum_{i=1}^{n} z_i^j t_i^+\kappa^{-1}\left\langle\prod_{\ell=1}^{n}\Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0 \qquad \text{for} \quad 0\leq j\leq r\,, \tag{2.32}$$

$$\left(\sum_{i=1}^{n} t_i^3+\frac{k}{2}r\right)\left\langle\prod_{\ell=1}^{n}\Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle = 0\,, \tag{2.33}$$

where $\mathcal{E}_{\{i_\alpha\}}$ is a combination of r ordinary KZ equations $\mathcal{E}_i$ (2.21),

$$\mathcal{E}_{\{i_\alpha\}} \equiv \sum_{\alpha=1}^{r}(t_{i_\alpha}^+)^{-1}\frac{1}{\prod_{\beta\neq\alpha}(z_{i_\alpha}-z_{i_\beta})}\mathcal{E}_{i_\alpha}\,, \tag{2.34}$$

and the last equation (2.33) is the J^3 part of the spacetime SL(2) symmetry. (The other parts are implicitly included in the previous equations.)

2.4 Sklyanin's separation of variables

The ordinary KZ equations, as well as the modified (combinations of) KZ equations for correlators of spectral-flowed fields, involve Lie algebra generators t_i^a acting on all the fields $i = 1,2,\ldots,n$. However, Sklyanin has shown how to separate them by a change of variables [6]. Since it maps the KZ equations to the Belavin-Polyakov-Zamolodchikov equations [7], this change of variables leads to a relation between correlators in the euclidean AdS_3 and correlators in Liouville theory [5]. In preparation for the extension of such a relation to correlators of spectral-flowed fields, I will now show how to perform the separation of variables in the equations (2.31)-(2.32).

I now assume that the field $\Phi^{j,w}(z)$ belongs to the principal continuous series $j \in -\frac{1}{2}+i\mathbb{R}$, and choose the μ-basis for this representation (see eq. (2.6)). This amounts to diagonalizing the operator t^+, with eigenvalue μ. This could be made explicit by using the notation $\Phi^{j,w}(z) = \Phi^{j,w}(\mu|z)$. Then the equation (2.32) simply becomes

$$u_j \equiv \sum_{\ell=1}^{n}\mu_\ell z_\ell^j = 0 \qquad \text{for} \quad 0\leq j\leq r\,. \tag{2.35}$$

Let me define new variables as the zeroes of the rational function

$$R(t) \equiv \sum_{\ell=1}^{n}\frac{\mu_\ell}{t-z_\ell}\,. \tag{2.36}$$

The number of zeroes of $R(t)$ is found by reducing it to the same denominator,

$$R(t) = \frac{\sum_{d=0}^{n-1}\left(\sum_{j=0}^{d} p_j u_{d-j}\right) t^{n-1-d}}{\prod_{\ell=1}^{n}(t-z_\ell)} \qquad \text{where} \qquad \prod_{\ell=1}^{n}(t-z_\ell) = \sum_{j=0}^{n} p_j t^{n-j} \quad \text{defines } p_j \,. \tag{2.37}$$

Since $u_j = 0$ for $0 \leq j \leq r$, the denominator of $R(t)$ has degree $n-2-r$. This defines the new variables y_a as

$$\sum_{\ell=1}^{n} \frac{\mu_\ell}{t-z_\ell} = u_{r+1} \frac{\prod_{a=1}^{n-2-r}(t-y_a)}{\prod_{\ell=1}^{n}(t-z_\ell)} \,. \tag{2.38}$$

Now I am in a position to perform the change of variables

$$\begin{array}{ccc} (\mu_1,\mu_2,\ldots,\mu_n)\big|_{u_0=u_1=\cdots=u_r=0} & & (y_1,y_2,\ldots,y_{n-2-r},u_{r+1}) \\ (n \text{ variables subject to } r+1 \text{ constraints}) & \rightarrow & (n-r-1 \text{ variables})\,. \end{array} \tag{2.39}$$

It is also convenient to perform a change of unknown function by explicitly solving the equations $u_0 = u_1 = \cdots = u_r = 0$ (2.32),

$$\left\langle \prod_{\ell=1}^{n} \Phi^{j_\ell, w_\ell}(\mu_\ell|z_\ell) \right\rangle = \kappa \prod_{j=0}^{r} \delta(u_j)\; \Omega_{n,r}\left(u_{r+1}, y_1, y_2, \cdots y_{n-2-r}|z_1, z_2, \cdots z_n\right). \tag{2.40}$$

Claim. *The system of linear combinations of KZ equations (2.31) satisfied by* $\kappa^{-1}\left\langle \prod_{\ell=1}^{n} \Phi^{j_\ell,w_\ell}(\mu_\ell|z_\ell)\right\rangle$*, which amounts to* $n+1-r$ *differential equations, is equivalent to* $\Omega_{n,r}$ *satisfying the* $n-2-r$ *BPZ equations characteristic of the Liouville correlator* $\left\langle \prod_{\ell=1}^{n} V_{\alpha_\ell}(z_\ell) \prod_{a=1}^{n-2-r} V_{-\frac{1}{2b}}(y_a)\right\rangle$

$$\left[b^2 \frac{\partial^2}{\partial y_a^2} + \sum_{a'\neq a}\left(\frac{1}{y_{aa'}}\frac{\partial}{\partial y_{a'}} + \frac{\Delta_{-\frac{1}{2b}}}{y_{aa'}^2}\right) + \sum_{i=1}^{n}\left(\frac{1}{y_a - z_i}\frac{\partial}{\partial z_i} + \frac{\Delta_{\alpha_i}}{(y_a-z_i)^2}\right)\right]\Theta_{n,r}^{\frac{2-k}{2}}\,\Omega_{n,r} = 0\,, \tag{2.41}$$

plus the three worldsheet SL(2) *equations*

$$\sum_{i=1}^{n} z_i^{0,1,2} \mathcal{E}_i\, \Omega_{n,r} = 0\,. \tag{2.42}$$

Notations: $b = (k-2)^{-\frac{1}{2}}$, $\alpha_i = b(j_i+1) + \frac{1}{2b}$, $\Delta_\alpha = \alpha(b+b^{-1}-\alpha)$, $\Delta_{-\frac{1}{2b}} = -\frac{1}{2} - \frac{3}{4b^2}$,

$$\Theta_{n,r} \equiv \frac{\prod_{i<i'\leq n} z_{ii'} \prod_{a<a'\leq n-2-r} y_{aa'}}{\prod_{i=1}^{n}\prod_{a=1}^{n-2-r}(z_i - y_a)}\,. \tag{2.43}$$

The rest of the subsection is devoted to proving this claim.

First, notice that $\Omega_{n,r}$ satisfies $\mathcal{E}_{\{i_\alpha\}}\Omega_{n,r} = 0$. This follows from the equations (2.31), (2.33) and from

$$[\mathcal{E}_{\{i_\alpha\}}, \prod_{j=0}^{r}\delta(u_j)] = 0 \qquad \text{modulo} \quad \sum_{\ell=1}^{n} t_\ell^3 + \frac{k}{2} r\,. \tag{2.44}$$

This can be proved by a direct if tedious computation.

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Then, rewrite the equations $\mathcal{E}_{\{i_\alpha\}}$ (2.34) as

$$\left(\sum_{i=1}^{n} \frac{\prod_{\ell\neq i_1,i_2,\ldots,i_r}(z_i-z_\ell)}{\prod_{a=1}^{n-2-r}(z_i-y_a)}\mathcal{E}_i\right)\Omega_{n,r}=0 \qquad \text{for} \quad \{i_1,i_2,\ldots,i_r\}\subset\{1,2,\ldots,n\}, \tag{2.45}$$

using $t_i^+ = \mu_i = u_{r+1}\frac{\prod_{a=1}^{n-2-r}(z_i-y_a)}{\prod_{\ell\neq i}(z_i-z_\ell)}$. Taking linear combinations of these equations for different choices of $\{i_1,i_2,\ldots,i_r\}$ yields the equivalent system

$$\left(\sum_{i=1}^{n} \frac{z_i^j}{\prod_{a=1}^{n-2-r}(z_i-y_a)}\mathcal{E}_i\right)\Omega_{n,r}=0\,, \qquad \text{for} \quad j=0,1,\ldots,n-r\,. \tag{2.46}$$

Further linear combinations of these equations lead to the worldsheet SL(2) equations (2.42) and to the equations

$$\sum_{i=1}^{n}\frac{1}{z_i-y_a}\mathcal{E}_i=0\,, \qquad \text{for} \quad a=1,2,\ldots,n-2-r\,. \tag{2.47}$$

Now these equations are equivalent to the BPZ equations (2.41), by the same argument as in the spectral flow-preserving case, see [5].

As a check, I computed the z-scaling behaviour of $\Omega_{n,r}$ by using the equation $\sum_{i=1}^n z_i\mathcal{E}_i\Omega_{n,r}=0$. The result agrees with the scaling

$$\sum_{i=1}^{n} z_i\frac{\partial}{\partial z_i}\,\kappa^{-1}\left\langle\prod_{\ell=1}^{n}\Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle=\left(-\sum_{i=1}^{n}\Delta_{j_i}-\frac{k}{4}r^2\right)\kappa^{-1}\left\langle\prod_{\ell=1}^{n}\Phi^{j_\ell,w_\ell}(z_\ell)\right\rangle\,, \tag{2.48}$$

which is expected from the conformal dimensions of the operators $\Phi^{j,w}_m$ (2.8).

3. Correlation functions with spectral-flowed states

Until now I have considered general properties of conformal field theories with a chiral $\widehat{s\ell_2}$ symmetry. In this section I plan to exploit these properties in the case of particular models. The most physically interesting model with $\widehat{s\ell_2}$ symmetry is string theory in AdS_3. However, this theory has a complicated spectrum including discrete states, and directly addressing it is difficult. Therefore, I will consider the euclidean version of that model, also known as the H_3^+ model. Although non unitary [8], this model has the advantages of being euclidean and of having a purely continuous spectrum.

My purpose is therefore to explicitly relate all correlation functions and conformal blocks of the H_3^+ model on a sphere to similar quantities in Liouville theory, a simpler non-rational conformal field theory. H_3^+ physical correlators were already related to Liouville theory in [5]; now I want to extend this relation to conformal blocks, and to correlators involving spectral-flowed fields. Such fields are unphysical in the H_3^+ model since they do not appear in the spectrum, but they play an important rôle in AdS_3 string theory.

In order to fully characterize H_3^+ correlators, the chiral results of the last section (namely the differential equations they satisfy) have to be supplemented with two types of information: how the left-moving and right-moving sectors are put together, and which structure constants appear in the operator product expansions. These data are already known, but in the next subsection I will recast them in a form which emphasizes the reflection symmetry and the relation to Liouville theory. Moreover, I will interpret them in terms of two alternative operator product expansions in the H_3^+ model. More details on these models can be found in [5] and references therein.

3.1 The H_3^+ model, Liouville theory, and their structure constants

3.1.1 The three-point function of the H_3^+ model

The H_3^+ model is a conformal field theory with symmetry algebra $\widehat{s\ell_2}\times\widehat{s\ell_2}$. The spectrum is made of physical fields $\Phi^j(z,\bar{z})$, $j\in -\frac{1}{2}+i\mathbb{R}$ transforming as vectors in the principal continuous series representation of spin j of both $\widehat{s\ell_2}$ algebras. Therefore, the physical fields transform as products $\Phi^j(z,\bar{z})\sim\Phi^j(z)\Phi^j(\bar{z})$ of the chiral fields of the previous section; however this chiral factorization fails at the level of the zero modes [8]. The spectral flowed fields $\Phi^{j,w\neq 0}(z,\bar{z})$ do not belong to the spectrum.

Two different bases for the spin j representation will appear: the μ-basis (see eq. (2.6)) and the m-basis, whose elements diagonalize the operators $t^+,\bar{t}^+$ and $t^3,\bar{t}^3$ respectively. They are related by

$$\Phi^{j,w}_{m,\bar{m}}(z,\bar{z})=N^j_{m,\bar{m}}\int\frac{d^2\mu}{|\mu|^2}\mu^m\bar{\mu}^{\bar{m}}\Phi^{j,w}(\mu,\bar{\mu}|z,\bar{z})\,,\qquad N^j_{m,\bar{m}}=\frac{\Gamma(-j-m)}{\Gamma(j+1+\bar{m})}\,. \tag{3.1}$$

(Note the change of convention $m\to -m$ wrt [5].) In later formulas, the antiholomorphic dependence on $\bar{z}$ and $\bar{\mu}$ may be omitted. The physical values of $m,\bar{m}$ obey $m-\bar{m}\in\mathbb{Z}$ and $m+\bar{m}\in i\mathbb{R}$.

The H_3^+ two-point function has to preserve spectral flow. Therefore, the flowed two-point function can be deduced from the unflowed one by using formula (2.24):

$$\begin{aligned}\left\langle\Phi^{j_1,w}(\mu_1|z_1)\Phi^{j_2,-w}(\mu_2|z_2)\right\rangle &= |z_{12}|^{-4\Delta_{j_1}+kw^2}|\mu_1|^2\delta^{(2)}(\mu_1+(-1)^w z_{12}^{2w}\mu_2)\times\\ &\quad\times\left[\delta(j_2+j_1+1)+R^H(j_1)\delta(j_2-j_1)\right], \end{aligned}\tag{3.2}$$

$$\begin{aligned}\left\langle\Phi^{j_1,w}_{m_1,\bar{m}_1}(z_1)\Phi^{j_2,-w}_{m_2,\bar{m}_2}(z_2)\right\rangle &= |z_{12}|^{-4\Delta^{j_1,w}_{m_1}}(-1)^{m_1-\bar{m}_1}\delta^{(2)}(m_1+m_2)\times\\ &\quad\times\left[\delta(j_2+j_1+1)+R^H(j_1)\frac{\Gamma(-j_1+m_1)}{\Gamma(j_1+1+m_1)}\frac{\Gamma(-j_1-\bar{m}_1)}{\Gamma(j_1+1-\bar{m}_1)}\times\right.\\ &\quad\left.\times\,\delta(j_2-j_1)\right], \end{aligned}\tag{3.3}$$

where $\Delta^{j_1,w}_{m_1}$ is defined in eq. (2.8), and, using the notation $b^2=\frac{1}{k-2}$,

$$R^H(j)=-\left(\frac{1}{\pi}b^2\gamma(b^2)\right)^{-(2j+1)}\frac{\Gamma(+2j+1)}{\Gamma(-2j-1)}\frac{\Gamma(+b^2(2j+1))}{\Gamma(-b^2(2j+1))}\,. \tag{3.4}$$

This H_3^+ reflection coefficient is actually identical to the Liouville theory reflection coefficient $R^L(\alpha)$, provided $b^2=\frac{1}{k-2}$ is interpreted as the usual parameter of Liouville theory (such that $c = 1 + 6Q^2$ with $Q = b + b^{-1}$), and the Liouville momentum α is given by [7, 5]

$$\alpha = b(j+1) + \frac{1}{2b} \,. \tag{3.5}$$

The H_3^+ three-point function can either preserve spectral flow or violate it by one unit. Let me start by recalling the spectral flow-preserving three-point function, while introducing new notations for the structure constants:

$$\left\langle \prod_{\ell=1}^3 \Phi^{j_\ell, w_\ell}(\mu_\ell|z_\ell) \right\rangle \underset{\sum w=0}{=} |z_{12}|^{-2\Delta_{12}^3 - kw_1w_2} |z_{13}|^{-2\Delta_{13}^2 - kw_1w_3} |z_{23}|^{-2\Delta_{23}^1 - kw_2w_3} \times \tag{3.6}$$
$$\times \delta^{(2)}(\mu_1\rho_1 + \mu_2\rho_2 + \mu_3\rho_3) D^H \begin{bmatrix} j_1 & j_2 & j_3 \\ \mu_1\rho_1 & \mu_2\rho_2 & \mu_3\rho_3 \end{bmatrix} C^H(j_1, j_2, j_3) \,,$$

with ρ_i is defined by (2.15) and $\Delta_{12}^3 = \Delta_{j_1} + \Delta_{j_2} - \Delta_{j_3}$. I define the structure constant C^H as

$$C^H(j_3, j_2, j_1) = -\frac{1}{2\pi^3 b} \left[\frac{\gamma(b^2) b^{2-2b^2}}{\pi}\right]^{-2-\Sigma j_i} \frac{\Upsilon_b'(0)}{\Upsilon_b(-b(j_{123}+1))\Gamma(-j_{123}-1)} \times$$
$$\times \frac{\Upsilon_b(-b(2j_1+1))\Upsilon_b(-b(2j_2+1))\Upsilon_b(-b(2j_3+1))}{\Upsilon_b(-bj_{12}^3)\Gamma(-j_{12}^3)\ \Upsilon_b(-bj_{13}^2)\Gamma(-j_{13}^2)\ \Upsilon_b(-bj_{23}^1)\Gamma(-j_{23}^1)} \,, \tag{3.7}$$

where $j_{12}^3 = j_1 + j_2 - j_3$ and $j_{123} = j_1 + j_2 + j_3$ and the definition of the special function Υ_b can be found in [5]. Notice the extra Γ factors with respect to the standard definition [9, 10]. They are added so that C^H is reflection-covariant like the three-point function itself,

$$C^H(j_1, j_2, j_3) = R^H(j_3) C^H(j_1, j_2, -j_3 - 1) \,. \tag{3.8}$$

Therefore, the factor D^H, and the three-point conformal block, are now reflection-invariant:

$$D^H \begin{bmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{bmatrix} = \pi \left|\frac{\mu_1}{\mu_2}\right|^{2j_1+2} |\mu_2|^2 \times \tag{3.9}$$
$$\times \sum_{\eta=\pm} \gamma_{j_3^\eta}^{j_1,j_2} \left|\frac{\mu_3}{\mu_2}\right|^{-2j_3^\eta} {}_2\mathcal{F}_1\left(j_1 - j_2 - j_3^\eta, j_1 + j_2 - j_3^\eta + 1, -2j_3^\eta, -\frac{\mu_3}{\mu_2}\right) ,$$
$$\gamma_{j_3}^{j_1,j_2} \equiv \Gamma(-j_{123}-1)\Gamma(-j_{23}^1)\Gamma(-j_{13}^2)\Gamma(j_{12}^3+1)\gamma(2j_3+1) \,, \tag{3.10}$$

where $j^+ = j$, $j^- = -j-1$, and ${}_2\mathcal{F}_1(a,b,c,z) = F(a,b,c,z)F(a,b,c,\bar{z})$. In this formula, the permutation symmetry of D^H is not manifest, indeed j_3 plays a priviledged rôle and the formula could be called a "j_3-decomposition" of D^H into two terms $\eta = \pm$. The other possible decompositions associated with j_1 and j_2 naturally give the same result; this is a consequence of the monodromy properties of the hypergeometric function ${}_2F_1$, which will shortly be interpreted as Liouville braiding.

Now let me consider the H_3^+ spectral flow-violating three-point function. This was determined in [11, 12]:

$$\left\langle \prod_{\ell=1}^3 \Phi_{m_\ell, \bar{m}_\ell}^{j_\ell, w_\ell}(z_\ell) \right\rangle \underset{\sum w=-1}{=} \left| z_{12}^{\Delta_{m_3}^{j_3,w_3} - \Delta_{m_1}^{j_1,w_1} - \Delta_{m_2}^{j_2,w_2}} z_{13}^{\Delta_{m_2}^{j_2,w_2} - \Delta_{m_1}^{j_1,w_1} - \Delta_{m_3}^{j_3,w_3}} z_{23}^{\Delta_{m_1}^{j_1,w_1} - \Delta_{m_2}^{j_2,w_2} - \Delta_{m_3}^{j_3,w_3}} \right|^2 \times$$

$$\times\delta^{(2)}\Big(\sum m_\ell - \frac{k}{2}\Big)\prod_{\ell=1}^{3} N^{j_\ell}_{m_\ell,\bar{m}_\ell} \times \tilde{C}^H(j_1,j_2,j_3)\,, \tag{3.11}$$

A useful notation. For conciseness, the modulus squared of m-dependent expressions means

$$|Y(z,m)|^2 = Y(z,m)\times Y(z\to\bar{z},\ m\to\bar{m})\,, \tag{3.12}$$

that is a product of factors depending on m and $\bar{m}$, although m and $\bar{m}$ are not complex conjugates.

The spectral flow-violating structure constant has been determined up to a k-dependent normalization:

$$\tilde{C}^H \simeq \frac{\left[\frac{1}{\pi}\gamma(b^2)b^{2-2b^2}\right]^{-j_{123}}}{\Upsilon_b(b(j_{123}+1+\frac{k}{2}))}\frac{\Upsilon_b(-b(2j_1+1))\Upsilon_b(-b(2j_2+1))\Upsilon_b(-b(2j_3+1))}{\Upsilon_b(b(j^3_{12}+\frac{k}{2}))\Upsilon_b(b(j^2_{13}+\frac{k}{2}))\Upsilon_b(b(j^1_{23}+\frac{k}{2}))}\,. \tag{3.13}$$

In the μ basis, the spectral flow-violating three-point function is:

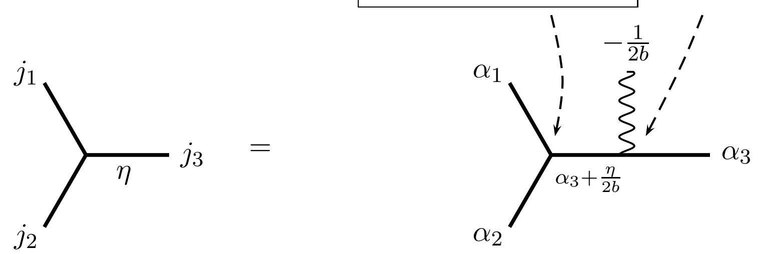
$$\begin{aligned}\left\langle\prod_{\ell=1}^{3}\Phi^{j_\ell,w_\ell}(\mu_\ell|z_\ell)\right\rangle_{\sum w=-1} = {} & |z_{12}|^{\frac{k}{2}-2-kw_1w_2-2\Delta^3_{12}}|z_{13}|^{\frac{k}{2}-2-kw_1w_3-2\Delta^2_{13}}|z_{23}|^{\frac{k}{2}-2-kw_2w_3-2\Delta^1_{23}}\times\\ & \times\frac{1}{4\pi^2}\delta^{(2)}\Big(\sum_{\ell=1}^{3}\mu_\ell\rho_\ell\Big)\delta^{(2)}\Big(\sum_{\ell=1}^{3}\mu_\ell\rho_\ell z_\ell\Big)\left|\sum_{\ell=1}^{3}\mu_\ell\rho_\ell z_\ell^2\right|^{4-k}\times\\ & \times\tilde{C}^H(j_1,j_2,j_3)\,.\end{aligned} \tag{3.14}$$

The μ-dependence in this formula is derived with the help of the results in section 2, in particular the equation (2.30). But of course the μ-basis result is equivalent to the previous m-basis formula.

3.1.2 Comparison with Liouville theory

The H_3^+ structure constants $C^H(j_1,j_2,j_3)$ and $\gamma^{j_1,j_2}_{j_3}$ are related to Liouville structure constants with momenta $\alpha = b(j+1)+\frac{1}{2b}$ as follows [5]:

$$C^H(j_1,j_2,j_3)\quad \gamma^{j_1,j_2}_{j_3^\eta} \quad = -\frac{2\pi^3}{b}\ \boxed{C^L\Big(\alpha_1,\alpha_2,\alpha_3+\frac{\eta}{2b}\Big)}\ \ \boxed{C^L_{-\eta}(\alpha_3)} \tag{3.15}$$



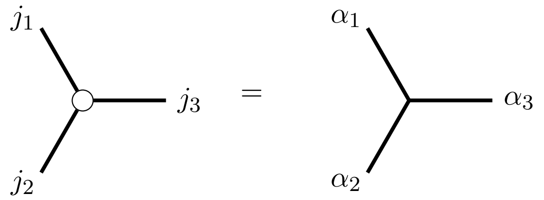
Here the diagrams illustrate relations between structure constants, which will later be promoted into relations between conformal blocks. These relations mean that the two terms $\eta = \pm$ of an H_3^+ vertex in the j_3-decomposition are equal to the two contributions to a Liouville vertex dressed with one degenerate field $\alpha = -\frac{1}{2b}$, associated with the two fusion channels $\alpha_3 \times -\frac{1}{2b} \to \alpha_3 + \frac{\eta}{2b}$. Fusing the degenerate field with α_1 or α_2 would yield the j_1- or j_2-decompositions of the H_3^+ vertex respectively. The different decompositions are therefore related by Liouville braiding as claimed above.

For completeness, let me recall the expressions [13, 14] for the Liouville structure constants which appear in eq. (3.15): (In the relation with the H_3^+ model, the Liouville interaction strength is fixed to $\mu_L = \frac{b^2}{\pi^2}$.)

$$\begin{aligned} C^L(\alpha_3,\alpha_2,\alpha_1) &= \frac{\left[\pi\mu_L\gamma(b^2)b^{2-2b^2}\right]^{b^{-1}(Q-\alpha_{123})}}{\Upsilon_b(\alpha_{123}-Q)} \frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_{12}^3)\Upsilon_b(\alpha_{23}^1)\Upsilon_b(\alpha_{31}^2)}\,, && (3.16) \\ C_-^L(\alpha) &= R^L(\alpha)R^L\left(Q-\alpha-\frac{1}{2b}\right), \qquad C_+^L(\alpha) = 1\,, && (3.17) \\ R^L(\alpha) &= (\pi\mu_L\gamma(b^2))^{\frac{Q-2\alpha}{b}} \frac{\Gamma(1+b(2\alpha-Q))}{\Gamma(1-b(2\alpha-Q))} \frac{\Gamma(1+b^{-1}(2\alpha-Q))}{\Gamma(1-b^{-1}(2\alpha-Q))}\,. && (3.18) \end{aligned}$$

The comparison of the spectral flow-violating structure constant $\tilde{C}^H$ with Liouville theory yields a very simple result: (The authors of [15] hint at such an equality but do not write it explicitly.)

$$\tilde{C}^H(j_1,j_2,j_3) \quad = \quad c_k\; C^L(\alpha_1,\alpha_2,\alpha_3)$$



(3.19)

Here c_k is a k-dependent constant, whose determination would require a more precise calculation of $\tilde{C}^H$.

3.1.3 The operator product expansion

Let me deduce the operator product expansion in the H_3^+ model from the two-point and three-point correlation functions. Using the two-point function (3.2), one can write:

$$\begin{aligned} \Phi^{j_1,w_1}(\mu_1|z_1)\Phi^{j_2,w_2}(\mu_2|z_2) \underset{z_{12}\to 0}{\sim} &\int_{-\frac{1}{2}+i\mathbb{R}} dj_s \int \frac{d^2\mu_s}{|\mu_s|^2}\, |z_{1s}|^{4\Delta_{j_s}-kw_s^2} \times \\ &\times \Big\langle \Phi^{j_1,w_1}(\mu_1|z_1)\Phi^{j_2,w_2}(\mu_2|z_2) \times \\ &\times \Phi^{-j_s-1,-w_s}(-(-1)^{w_s} z_{1s}^{-2w_s}\mu_s|z_s)\Big\rangle \Phi^{j_s,w_s}(\mu_s|z_1)\,. \end{aligned} \tag{3.20}$$

Here, z_s is an auxiliary worldsheet coordinate which disappears as $z_{12}\to 0$.

It is now necessary to discuss the domain of validity of the OPE eq. (3.20), and in particular the value of w_s. When there is no spectral flow, $w_1 = w_2 = 0$, it is known [10] that the OPE holds for $w_s = 0$: the OPE of unflowed fields yields only unflowed fields. The natural generalization is: the OPE preserves spectral flow, and eq. (3.20) holds for

$w_s = w_1 + w_2$. However, the ordinary OPE $\Phi^{j_1,0}\Phi^{j_2,0} \sim \int dj_s\, \Phi^{j_s,0}$ may not hold when the fields are applied to states outside the physical spectrum, like states created by spectral flowed fields. For instance, using this OPE to compute a flow-violating three-point function $\left\langle \Phi^{j_1,0}\Phi^{j_2,0}\Phi^{j_3,-1}\right\rangle$ would not yield the right result. The correct result can however be obtained by using the OPE eq. (3.20) with $w_s = w_1 + w_2 + 1$.

In string theory in the minkowskian AdS_3, all values of the spectral flow number w_s appear in the spectrum and the OPE is expected to be of the form, $\Phi^{j_1,w_1}\Phi^{j_2,w_2} \sim \int dj_s \sum_{|w_s-w_1-w_2|\leq 1} \Phi^{j_s,w_s}$. This cannot happen in the H_3^+ model, because the spectrum is much smaller. Indeed, according to [10], the H_3^+ four-point function $\left\langle \Phi^{j_1,0}\Phi^{j_2,0}\Phi^{j_3,0}\Phi^{j_4,0}\right\rangle$ can be decomposed by using the ordinary, flow-preserving OPE. No extra terms of the type $\left\langle \Phi^{j_s,1}\Phi^{j_3,0}\Phi^{j_4,0}\right\rangle$ appear.

There are also cases, like the correlator $\left\langle \Phi^{j_1,0}\Phi^{j_2,0}\Phi^{j_3,0}\Phi^{j_4,-1}\right\rangle$ with one unit of spectral flow violation, where both OPEs $\Phi^{j_1,0}\Phi^{j_2,0} \sim \int dj_s\ \Phi^{j_s,0}$ and $\Phi^{j_1,0}\Phi^{j_2,0} \sim \int dj_s\ \Phi^{j_s,1}$ can be used and yield the same result for the correlator in question. This will be demonstrated in the next subsection, and is evidence for the following hypothesis:

Hypothesis. *The H_3^+ operator product expansion eq. (3.20) can hold with either $w_s = w_1 + w_2 - 1$ or $w_s = w_1 + w_2$ or $w_s = w_1 + w_2 + 1$, depending on which correlator the expansion is inserted in. These possibilities are not exclusive, in particular both $w_s = w_1 + w_2$ and $w_s = w_1 + w_2 + 1$ expansions can be used in a correlator with spectral flow violation $0 < r < n - 2$.*

To prove this hypothesis would require a study of the fields $\Phi^{j,w}$ as differential operators and of the domains they act on, which I shall not attempt. Further evidence will however come with the result eq. (3.26) for the H_3^+ correlators, derived using the hypothesis, and which is compatible with both choices of OPEs $w_s = w_1 + w_2$ and $w_s = w_1 + w_2 + 1$, when such a choice is available.

Moreover, the hypothesis is consistent with the definition of spectral flowed correlators reported in [12, 15] and attributed to Fateev, Zamolodchikov and Zamolodchikov. This definition indeed involves the insertion of the H_3^+ operator of spin $-\frac{k}{2}$:

$$\left\langle \Phi^{j_1,1}(z_1)\Phi^{j_2,0}(z_2)\cdots\right\rangle \propto \lim_{u\to z_1} \left\langle \Phi^{j_1}(z_1)\Phi^{j_2}(z_2)\Phi^{-\frac{k}{2}}(u)\cdots\right\rangle \propto \left\langle \Phi^{-j_1-\frac{k}{2}}(z_1)\Phi^{j_2}(z_2)\cdots\right\rangle . \tag{3.21}$$

In the limit $z_1 \to z_2$, this leads to the flow-violating OPE:

$$\lim_{z_1\to z_2} \left\langle \Phi^{j_1,1}(z_1)\Phi^{j_2,0}(z_2)\cdots\right\rangle \propto \lim_{z_1\to z_2} \left\langle \Phi^{-j_1-\frac{k}{2}}(z_1)\Phi^{j_2}(z_2)\cdots\right\rangle \propto \int dj_s\ \left\langle \Phi^{j_s,0}\cdots\right\rangle , \tag{3.22}$$

where the last operation was an ordinary H_3^+ OPE which yielded an unflowed field. However inverting the limits $z_1 \to z_2$ and $u \to z_1$ leads to

$$\begin{aligned} \lim_{u\to z_1}\lim_{z_1\to z_2} \left\langle \Phi^{j_1}(z_1)\Phi^{j_2}(z_2)\Phi^{-\frac{k}{2}}(u)\cdots\right\rangle &\propto \lim_{u\to z_1} \int dj_s\ \left\langle \Phi^{j_s}(z_1)\Phi^{-\frac{k}{2}}(u)\cdots\right\rangle \\ &\propto \int dj_s\ \left\langle \Phi^{j_s,1}(z_1)\cdots\right\rangle , \end{aligned} \tag{3.23}$$

i.e. a spectral flow-preserving OPE.

The hypothesis above suggests that the states $|j, w \neq 0\rangle$ created by spectral-flowed operators in the H_3^+ model are similar to the states $|j \notin -\frac{1}{2} + i\mathbb{R}, 0\rangle$: they do not belong to the physical spectrum and do not appear in the physical OPE, but they have a non-vanishing three-point function with physical states, which can be accounted for by a non-physical OPE. However, in contrast to the flow-violating OPE, the non-phyisical OPE involving $j \notin -\frac{1}{2} + i\mathbb{R}$ is obtained from the physical OPE by deforming the contour of integration $\int_{-\frac{1}{2}+i\mathbb{R}} dj_s$.

Now here are explicit expressions for the OPEs derived from eq. (3.20), obtained by inserting the explicit expression for the three-point function. This yields the spectral flow-preserving OPE in the case $w_s = w_1 + w_2$,

$$\begin{aligned}\Phi^{j_1,w_1}(\mu_1|z_1)\Phi^{j_2,w_2}(\mu_2|z_2) \underset{z_{12}\to 0}{\sim} & \int_{-\frac{1}{2}+i\mathbb{R}} dj_s\ C^H(j_1,j_2,-j_s-1)|z_{12}|^{-2\Delta^s_{12}-kw_1w_2} \times \\ & \times |\mu_s|^{-2} D^H \begin{bmatrix} j_1 & j_2 & -j_s-1 \\ \mu_1 z_{12}^{w_2} & \mu_2 z_{21}^{w_1} & -\mu_s \end{bmatrix} \times \\ & \times \Phi^{j_s,w_1+w_2}(\mu_s = \mu_1 z_{12}^{w_2} + \mu_2 z_{21}^{w_1}|z_1)\,, \end{aligned} \tag{3.24}$$

and the spectral flow-violating OPE in the case $w_s = w_1 + w_2 + 1$,

$$\begin{aligned}\Phi^{j_1,w_1}(\mu_1|z_1)\Phi^{j_2,w_2}(\mu_2|z_2) \underset{z_{12}\to 0}{\sim} & \frac{1}{4\pi^2}\delta^{(2)}\left(\mu_1 z_{12}^{w_2+1} - \mu_2 z_{21}^{w_1+1}\right) \left|\mu_1 z_{12}^{w_2+1}\right|^{2-k} \times \\ & \times \int_{-\frac{1}{2}+i\mathbb{R}} dj_s \tilde{C}^H(j_1,j_2,-j_s-1)|z_{12}|^{-2\Delta^s_{12}+\frac{k}{2}-kw_1w_2} \times \\ & \times \Phi^{j_s,w_1+w_2+1}(\mu_1 z_{12}^{w_2+1}|z_1)\,. \end{aligned} \tag{3.25}$$

3.2 H_3^+ correlators from Liouville theory

3.2.1 Results

The relations between the H_3^+ model and Liouville theory at the levels of structure constants eq. (3.15), (3.19), and differential equations reflecting chiral symmetry eq. (2.41), lead to the following expression for the H_3^+ correlation functions:

$$\boxed{\begin{aligned}\left\langle \prod_{\ell=1}^n \Phi^{j_\ell,w_\ell}(\mu_\ell|z_\ell)\right\rangle \underset{\sum w = -r \leq 0}{=} & \frac{\pi}{2}(-\pi)^{-n} b\ c_k^r \times \\ & \times \prod_{j=0}^r \delta^{(2)}\big(\textstyle\sum \mu_\ell \rho_\ell z_\ell^j\big) \left|\textstyle\sum \mu_\ell \rho_\ell z_\ell^{r+1}\right|^{2+2r-kr} \times \\ & \times |\Theta_{n,r}|^{k-2} \left\langle \prod_{\ell=1}^n V_{\alpha_\ell}(z_\ell) \prod_{a=1}^{n-2-r} V_{-\frac{1}{2b}}(y_a)\right\rangle \end{aligned}} \tag{3.26}$$

Let me recall the notations involved in this formula: $V_\alpha(z)$ is the Liouville vertex operator of conformal weight $\Delta_\alpha = \alpha(b + b^{-1} - \alpha)$, where the Liouville parameter is $b = (k-2)^{-\frac{1}{2}}$ and the interaction strength is $\mu_L = b^2/\pi^2$; the Liouville momenta $\alpha = b(j+1) + \frac{1}{2b}$ are such that $\Delta_\alpha = \Delta_j + \frac{k}{4}$; and the positions of the Liouville degenerate fields y_a are defined by

$$\sum_{\ell=1}^{n} \frac{\mu_\ell \rho_\ell}{t - z_\ell} = \left(\sum_{\ell=1}^{n} \mu_\ell \rho_\ell z_\ell^{r+1} \right) \frac{\prod_{a=1}^{n-2-r} (t - y_a)}{\prod_{\ell=1}^{n} (t - z_\ell)} \qquad \text{with} \quad \rho_\ell = \prod_{j \neq \ell} z_{\ell j}^{w_j} \,. \tag{3.27}$$

The factor $\Theta_{n,r}$ was defined in eq. (2.43), and the k-dependent factor c_k is not known. (c_k is related but not equal to the c_k of eq. (3.19); other unknown c_ks will appear below.)

The m-basis H_3^+ correlators are related to Liouville correlators by applying the change of basis (3.1) and the change of variables (2.39), whose jacobian is:

$$\prod_{i=1}^{n} \frac{d^2\mu_i}{|\mu_i|^2} \delta^{(2)} \left(\sum \mu_i z_i^r \right) \cdots \delta^{(2)} \left(\sum \mu_i \right) = \frac{d^2 u}{|u|^{4+2r}} \prod_{a=1}^{n-2-r} d^2 y_a \, \frac{\prod_{a<a'} |y_{aa'}|^2 \prod_{i<i'} |z_{ii'}|^2}{\prod_i \prod_a |y_a - z_i|^2} \,. \tag{3.28}$$

The result is:

$$\boxed{\begin{aligned} \left\langle \prod_{\ell=1}^{n} \Phi^{j_\ell, w_\ell}_{m_\ell, \bar{m}_\ell}(z_\ell) \right\rangle \underset{\sum w = -r \leq 0}{=} & \frac{2\pi^{3-2n} b \; c_k^r}{(n-2-r)!} \prod_{\ell=1}^{n} N^{j_\ell}_{m_\ell, \bar{m}_\ell} \times \delta^{(2)} \left(\sum m_\ell - \frac{k}{2} r \right) \times \\ & \times \left| \prod_{\ell<\ell'} z_{\ell\ell'}^{\beta_{\ell\ell'}} \right|^2 \int \prod_{a=1}^{n-2-r} d^2 y_a \, \frac{\prod_{a<a'} |y_{aa'}|^k}{\left| \prod_{\ell,a} (z_\ell - y_a)^{\frac{k}{2} - m_\ell} \right|^2} \times \\ & \times \left\langle \prod_{\ell=1}^{n} V_{\alpha_\ell}(z_\ell) \prod_{a=1}^{n-2-r} V_{-\frac{1}{2b}}(y_a) \right\rangle \end{aligned}} \tag{3.29}$$

where the combinatorial factor $\frac{1}{(n-2-r)!}$ comes from the invariance of the μ_ℓ wrt permutations of the y_as, and the exponent $\beta_{\ell\ell'}$ is defined by

$$\beta_{i\ell} \equiv \frac{k}{2} - \frac{k}{2} w_i w_\ell - w_i m_\ell - w_\ell m_i - m_i - m_\ell \,. \tag{3.30}$$

The formula (3.29) can be rephrased in the language of the parafermions (2.9) and it gives the n-point function $\left\langle \prod_{\ell=1}^{n} \Psi^{j_\ell}_{m_\ell, \bar{m}_\ell} \right\rangle$ provided $\beta_{i\ell}$ is replaced with

$$\beta'_{i\ell} = \frac{2}{k} \left(m_i - \frac{k}{2} \right) \left(m_\ell - \frac{k}{2} \right) . \tag{3.31}$$

The resulting expression for the parafermionic correlators agrees with the unpublished results of Fateev [16], obtained by free field methods.

The integrals over y_a in eq. (3.29) may have singularities at $y_a = z_i$, depending on the values of $m_\ell, \bar{m}_\ell$. The physical values in the H_3^+ model are $m - \bar{m} \in \mathbb{Z}$, $m + \bar{m} \in i\mathbb{R}$ and would make the integrals converge, but they are forbidden by the constraints $\sum m = \sum \bar{m} = \frac{k}{2} r$. However, assuming the external spins are physical $j_\ell \in -\frac{1}{2} + i\mathbb{R}$, the integrals actually converge provided $\mathrm{Re}(m_\ell + \bar{m}_\ell) > -1$. This mild condition from the point of view of the H_3^+ model becomes a problem when Wick-rotating to string theory in AdS_3, whose physical spectrum satisfies $m_\ell + \bar{m}_\ell \in \mathbb{R}$.

3.2.2 Arguments

Here I argue for (and partly prove) the relation between H_3^+ correlators in the μ basis and Liouville theory correlators, eq. (3.26). In the case when there is no spectral flow $w_\ell = 0$, this has been done in [5]. Then, in the spectral flow-preserving case $\sum w_\ell = -r = 0$, this is a simple consequence of the formula (2.24), where the action of the differential operator κ (2.22) on μ_ℓ accounts for the ρ_ℓ factors.

Now in the general case, it is possible to use an argument similar to the one in [5] in order to reduce the problem to maximally flow-violating correlators: whenever spectral flow violation is not maximal, it is possible to reduce the number of fields by using the flow-preserving OPE and the correspondence between KZ and BPZ equations of section 2.

However, I can only conjecture the validity of eq. (3.26) in the maximally flow-violating case. This is because the corresponding Liouville correlator involves no degenerate field and thus satisfies no BPZ equation. The z-dependence of that Liouville correlator is therefore not controlled by the techniques used so far. Nevertheless, I will present some strong evidence in favour of that conjecture. This comes in addition to the calculations of Fateev [16] that eq. (3.29) holds whenever both sides are accessible to free field computations.

First, the proposed relation with Liouville theory eq. (3.26) is compatible with the spectral flow-violating OPE eq. (3.25). The comparison between the spectral flow-violating structure constant $\tilde{C}^H$ and Liouville theory (3.19) shows that the OPE coefficients agree. Now consider the quantities $u_j \equiv \sum_{\ell=0}^n \mu_\ell \rho_\ell z_\ell^j$ appearing in an n-point function, and

$$u_j' \equiv \mu_1 z_{12}^{w_2+1} \prod_{\ell\geq 3} z_{1\ell}^{w_\ell} + \sum_{i\geq 3} \mu_i z_{i1}^{w_1+w_2+1} \prod_{\substack{\ell\geq 3\\ \ell\neq i}} z_{i\ell}^{w_\ell} , \tag{3.32}$$

which appears in the $n-1$-point function obtained by a spectral flow-violating OPE. Direct computation leads to

$$u_j' \underset{z_{12}\to 0}{\sim} u_{j+1} - z_1 u_j . \tag{3.33}$$

This equation is the key to showing that the positions y_a of the $n-2-r$ auxiliary fields are not affected by the OPE, and that the delta-function factors behave correctly:

$$\begin{aligned}\delta^{(2)}\left(\mu_1 z_{12}^{w_2+1} - \mu_2 z_{21}^{w_1+1}\right) \prod_{j=0}^{r-1} \delta^{(2)}(u_j') &\underset{z_{12}\to 0}{\sim} \delta^{(2)}(z_{12}u_0) \prod_{j=0}^{r-1} \delta^{(2)}(u_{j+1} - z_1 u_j) \\ &= |z_{12}|^{-2} \prod_{j=0}^{r} \delta^{(2)}(u_j) .\end{aligned} \tag{3.34}$$

These manipulations with δ-functions will not be very rigorous as long as I do not define the domain of these distributions. An alternative is to prove the equivalent m-basis result (3.29) instead of the μ-basis result. This is actually quite straightforward, but I have given the argument in the μ-basis in order to illustrate the μ-basis OPE. One reason to insist on the use of the μ-basis is that conformal blocks in this basis are much simpler than in any other basis, as will be demonstrated in the next subsection.

The second line of evidence in favour of the relation (3.26) in the maximally flow-violating case comes from the study of the special values of α and j where the correlators satisfy differential equations. On the Liouville side, this happens if α belongs to the Kac table associated with the central charge $c = 1 + 6Q^2$. The corresponding values of j turn out to give $\widehat{s\ell}_2$ representations with affine null vectors. Further evidence could be obtained by comparing the null vector differential equations themselves, but this is beyond the scope of the present article.

3.3 H_3^+ conformal blocks from Liouville theory

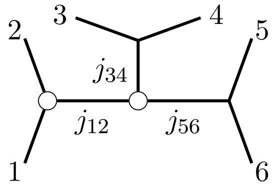
The relation (3.26) between H_3^+ correlators and Liouville correlators can be decomposed into relations between the structure constants of the theories, which I already wrote (3.15), (3.19), and relations between the conformal blocks.

Consider an n-point correlator in H_3^+ with r units of spectral flow violation, $\sum w = -r \leq 0$. I will consider decomposition in conformal blocks which use vertices with winding violation 0 or -1. If vertices with winding violation $+1$ were included, there would probably be no simple relation to Liouville theory. Moreover, for ease of writing I will only consider a specific case $n = 6$, $r = 2$, which involves $n - 2 - r = 2$ Liouville degenerate field insertions.

A basis of H_3^+ non-chiral conformal blocks is defined as follows:

$$\left\langle \prod_{\ell=1}^{6} \Phi^{j_\ell, w_\ell}(\mu_\ell|z_\ell) \right\rangle = \int_{-\frac{1}{2}+i\mathbb{R}} dj_{12}\ dj_{34}\ dj_{56}\ \tilde{C}^H(j_1, j_2, -j_{12} - 1)\ C^H(j_3, j_4, -j_{34} - 1) \times$$

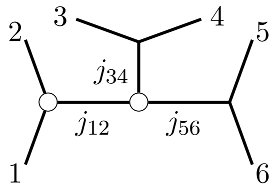
$$\times C^H(j_5, j_6, -j_{56} - 1) \tilde{C}^H(j_{12}, j_{34}, j_{56})$$



$$. \quad (3.35)$$

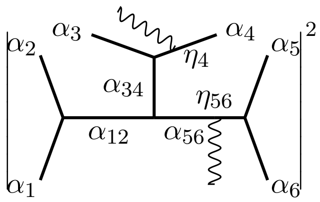
The pictorial representation for the conformal block leaves its dependence on $\mu_\ell, z_\ell, j_\ell, w_\ell$ implicit. Note that with our definition of the structure constant C^H (3.7), the conformal block is invariant wrt reflection of external spins $j_1 \cdots j_6$ and internal spins j_{12}, j_{34}, j_{56}.

This H_3^+ non-chiral conformal block can now be decomposed into Liouville chiral conformal blocks in the following way:



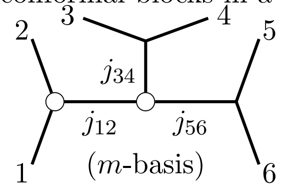
$$= \frac{\pi}{2}(-\pi)^{-n} b c_k^r \prod_{j=0}^{r} \delta^{(2)}\left(\sum \mu_\ell \rho_\ell z_\ell^j\right) \left|\sum \mu_\ell \rho_\ell z_\ell^{r+1}\right|^{2+2r-kr} \times \quad (3.36)$$

$$\times |\Theta_{n,r}|^{k-2} \sum_{\eta_4 = \pm} \gamma_{j_4^{\eta_4}}^{j_3, j_{34}} \sum_{\eta_{56} = \pm} \gamma_{j_{56}^{\eta_{56}}}^{j_5, j_6}$$



$$,$$

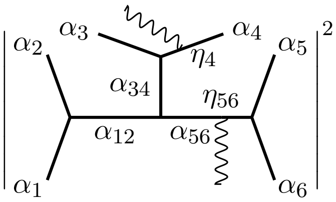
where the indices η_4, η_{56} indicate the fusion channels of the two degenerate fields, $\alpha_4 + \frac{\eta_4}{2b}$ and $\alpha_{56} + \frac{\eta_{56}}{2b}$, and the z_ℓ, y_a dependence of the Liouville conformal block on the positions of the fields is omitted. Alternative positionings of the degenerate field insertions are possible, as long as they remain around the Liouville vertices which correspond to the spectral flow-preserving vertices in H_3^+. Naturally, H_3^+ conformal blocks in the m-basis (and thus parafermionic conformal blocks) can also be expressed in terms of Liouville theory conformal blocks in a similar manner:



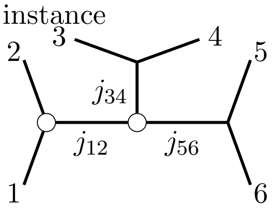
$$= \frac{2\pi^{3-2n} b\ c_k^r}{(n-2-r)!} \prod_{\ell=1}^{n} N^{j_\ell}_{m_\ell,\bar{m}_\ell} \times \delta^{(2)}\Big(\sum m_\ell - \frac{k}{2} r\Big) \Big| \prod_{\ell<\ell'} z_{\ell\ell'}^{\beta_{\ell\ell'}} \Big|^2 \times$$

$$\times \sum_{\eta_4=\pm} \gamma^{j_3,j_{34}}_{j_4^{;\eta_4}} \sum_{\eta_{56}=\pm} \gamma^{j_5,j_6}_{j_{56}^{;\eta_{56}}} \int \prod_{a=1}^{n-2-r} d^2 y_a \frac{\prod_{a<a'} |y_{aa'}|^k}{\Big| \prod_{\ell,a} (z_\ell - y_a)^{\frac{k}{2} - m_\ell} \Big|^2} \times$$

$$\times \Big| \quad \Big|^2 . \qquad (3.37)$$



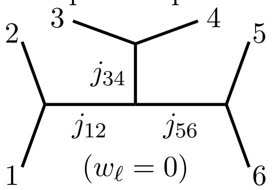
What is however unique to the μ-basis is the possibility of explicitly writing the H_3^+ conformal blocks in some limits, obtained by performing multiple OPEs (3.24), (3.25), for instance



$$\underset{z_{12},z_{34} \ll z_{13} \ll z_{15} \ll z_{56}}{\sim} \frac{1}{16\pi^4} \delta^{(2)}(\mu_1 z_{12}^{w_2+1} - \mu_2 z_{21}^{w_1+1}) \times$$

$$\times \delta^{(2)}(\mu_{12} z_{13}^{w_3+w_4+1} - \mu_{34} z_{31}^{w_1+w_2+2}) \delta^{(2)}(\mu_{12} z_{13}^{w_3+w_4+1} z_{15}^{w_5} + \mu_5 z_{51}^{-w_5-w_6} + \mu_6 z_{61}^{-2w_6}) \times$$
$$\times |\mu_{12}|^{2-2k} |z_{12}|^{-2\Delta_{1,2}^{12} + \frac{k}{2} - k w_1 w_2} |z_{34}|^{-2\Delta_{3,4}^{34} - k w_3 w_4} |z_{16}|^{-4\Delta_6 + k w_6^2} \times$$
$$\times |z_{13}|^{-2\Delta_{12,34}^{56} + 4 + 2(w_1+w_2) - k(w_3+w_4+1) + \frac{k}{2} - k(w_3+w_4)(w_1+w_2+1)} |z_{15}|^{-2\Delta_{56,5}^{6} + k w_5 (w_5+w_6)} \times$$
$$\times D^H \begin{bmatrix} j_3 & j_4 & j_{34} \\ \mu_3 z_{34}^{w_4} & \mu_4 z_{43}^{w_3} & -\mu_{34} \end{bmatrix} D^H \begin{bmatrix} j_{56} & j_5 & j_6 \\ \mu_{12} z_{13}^{w_3+w_4+1} z_{15}^{w_5} & \mu_5 z_{51}^{-w_5-w_6} & \mu_6 z_{61}^{-2w_6} \end{bmatrix}, \qquad (3.38)$$

where $\mu_{34} = \mu_3 z_{34}^{w_4} + \mu_4 z_{43}^{w_3}$ and $\mu_{12} = \mu_1 z_{12}^{w_2+1}$. In the case where all spectral flow numbers w_ℓ vanish, the conformal blocks reduce in such limits to $\mathrm{SL}(2,\mathbb{C})$ coinvariants which have an explicit expression as products of the D^H coefficients:



$$\underset{z_{12},z_{34} \ll z_{13} \ll z_{15} \ll z_{56}}{\sim} |z_{12}|^{-2\Delta_{1,2}^{12}} |z_{34}|^{-2\Delta_{3,4}^{34}} |z_{13}|^{-2\Delta_{12,34}^{56}} |z_{15}|^{-2\Delta_{56,5}^{6}} \times$$

$$\times |z_{16}|^{-4\Delta_6} \delta^{(2)}\Big(\sum \mu_\ell\Big) |\mu_{12}|^{-2} |\mu_{34}|^{-2} |\mu_{56}|^{-2} \times$$
$$\times D^H \begin{bmatrix} j_1 & j_2 & j_{12} \\ \mu_1 & \mu_2 & -\mu_{12} \end{bmatrix} D^H \begin{bmatrix} j_3 & j_4 & j_{34} \\ \mu_3 & \mu_4 & -\mu_{34} \end{bmatrix} \times$$
$$\times D^H \begin{bmatrix} j_5 & j_6 & j_{56} \\ \mu_5 & \mu_6 & -\mu_{56} \end{bmatrix} D^H \begin{bmatrix} j_{12} & j_{34} & j_{56} \\ \mu_{12} & \mu_{34} & \mu_{56} \end{bmatrix}, \qquad (3.39)$$

where $\mu_{\ell\ell'} = \mu_\ell + \mu_{\ell'}$. The H_3^+ flow-preserving conformal blocks are fully determined by their behaviour in such limits, plus the KZ equations.

4. Outlook

The present results may hopefully be useful in the study of string theory in AdS_3. Correlators in this theory should be obtained from H_3^+ correlators in the m basis (3.29) by Wick-rotation of $m_\ell + \bar{m}_\ell$, in the spirit of [12]. The integrals in eq. (3.29) then become divergent, and regularizing them should lead to the appearance of discrete states in the intermediate channels, even if all external states are continuous.

The expression (3.36) of H_3^+ conformal blocks in terms of Liouville theory conformal blocks suggests a way to define and compute the fusing matrix in the H_3^+ model. This fusing matrix is still not well understood, see [17]. However, knowing all H_3^+ correlators in terms of Liouville correlators makes this issue less crucial for the H_3^+ model proper. Nevertheless, an H_3^+ fusing matrix should be a very interesting object in itself, which may have an interpretation in terms of harmonic analysis on the quantum group $U_q(s\ell_2)$.

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